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MODULATIONAL STABILITY OF PERIODIC SOLUTIONS OF THE KURAMOTO-SIVASHINSKY EQUATION

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ABSTRACT

We study the long-wave, modulational, stability of steady periodic solutions of the Kuramoto-Sivashinsky equation. The analysis is fully nonlinear at first, and can in principle be carried out to all orders in the small parameter, which is the ratio of the spatial period to a characteristic length of the envelope perturbations. In the linearized regime we recover a high-order version of the results of Frisch, She and Thual, [1], which shows that the periodic waves are much more stable than previously expected.

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1 Introduction

The Kuramoto Sivashinsky (KS) equation

$$\begin{gathered} u_t + uu_x + u_{xx} + \nu u_{xxxx} = 0, \\ (x,t) \in \mathbf{R}^1 \times \mathbf{R}^+, \\ u(x,0) = u_0(x), \end{gathered} \tag{1.1}$$

with ν a positive parameter, arises as an amplitude equation in long-wave, weakly nonlinear stability analysis, in a variety of applications. It arises, for example, in free surface flow of viscous liquids [2, 3, 4, 5], in concentration waves in chemically reacting systems [6, 7, 8], in ion diffusion in plasmas [9], in flame propagation [10, 11] and in the dynamics of interfaces in two-phase flows [12, 13, 14]. It is the simplest equation with a convective nonlinearity and a band of unstable linear modes and therefore a good example on which to apply the general notions of inertial manifold theory. This means that the long time dynamics of KS is captured well by a finite dimensional dynamical system whose number of degrees of freedom is at least as large as the number of linearly unstable modes [15]-[19]. For 2π periodic solutions the number of linearly unstable modes is of order $\nu^{-1/2}$ while the best estimate for the dimension of the inertial manifold is $const \cdot \nu^{-21/40}$, [16], which is quite good except that the constant in the estimate is large.

The KS equation has been studied numerically by many authors [20]-[25]. The main objective of the numerical studies is to observe the appearance of low-dimensional dynamic behavior, including complicated time-periodic and chaotic behavior. Feigenbaum, [26, 27], and others [28, 29], have shown that period doubling and transition to chaos in the iteration of one dimensional maps have a universal behavior. This universal behavior is expected also in bifurcation of periodic solutions in low-dimensional dynamical systems, for example the Lorenz system [30]. Numerical computations [31] have verified this Feigenbaum period doubling route to chaos for this system. It is reasonable to expect then that partial differential equations whose large time behavior is captured by finite dimensional dynamical systems should also exhibit universal period doubling behavior. This was first seen numerically for the Ginzburg-Landau equation in [32], for thermal convection in [33] and for the KS equation in [34, 35]. Other aspects of the inertial manifold as its dimension increases are explored numerically in [36, 37].

When $\nu > 1$ there are no nontrivial steady, 2π periodic solutions of KS. For ν just below one there exist steady periodic solutions [38]. Steady solutions with Dirichlet boundary conditions and their stability were studied by Novick-Cohen [40]. In this paper we will

analyze the modulational stability of steady 2π periodic solutions $U(y;\nu)$ normalized to have mean zero

$$UU_y + U_{yy} + \nu U_{yyyy} = 0, \tag{1.2}$$
$$\overline{U} = \frac{1}{2\pi}\int_0^{2\pi} U(y)dy = 0 \quad , \quad U(y) = U(y+2\pi).$$

For any constants ρ and c the KS equation (1.1) has the two-parameter family of solutions

$$u(t,x) = \rho U(\rho(x-ct);\rho^2\nu) + c. \tag{1.3}$$

which are due to scale and Gallilean invariance of the KS equation. By modulational stability we mean the construction and analysis of solutions of KS in all of $\mathbf{R}$ near this family, when ρ and c are slowly varying with respect to the period of the steady solution (1.2). In [1], the linear modulational stability of U is analyzed by homogenization methods [41]. If ϵ is the ratio of the period 2π to the (long) scale of spatial modulations we must then study the behavior of solutions to

$$u_t + \frac{1}{\epsilon}U(\frac{x}{\epsilon})u_x + u_{xx} + \epsilon^2\nu u_{xxxx} = 0 \tag{1.4}$$

as ϵ tends to zero, which is a homogenization problem. This analysis was carried up to second order in ϵ in [1] and yields an effective diffusion equation with ν dependent diffusion coefficient which may be positive, corresponding to stable modulations, or negative, corresponding to unstable modulations. However, the stabilizing term u_{xxxx} does not influence the diffusion coefficient since it comes in at a higher order in the expansion. It is not surprising therefore that the stable range of ν obtained in [1] is very small.

In this paper we analyze the nonlinear modulational stability of U to sufficiently high order that the stabilizing term u_{xxxx} is accounted for fully. We use a method similar to the one in [42]. In addition to the expansion method, our main result is the determination of a much larger range of ν than that found in [1] for which there is stability.

2 Formulation.

We will construct modulated solutions of KS based on the family of solutions (1.3). Let

$$u(t,x) = \frac{\rho}{\epsilon}U(\frac{\theta}{\epsilon};\rho^2\nu) + \frac{c}{\epsilon} + w(\frac{\theta}{\epsilon},t,x). \tag{2.1}$$

where $\rho(t,x)$, $c(t,x)$, $\theta_x(t,x) = \rho(t,x)$ and $w(y,x,t)$, which is periodic in the y variable and has mean zero, are to be such that for ϵ sufficiently small u solves the suitably scaled KS equation

$$\frac{1}{\epsilon}u_t + uu_x + u_{xx} + \epsilon^2\nu u_{xxxx} = 0, \tag{2.2}$$

In (2.1), ϵ is the ratio of the period 2π to a characteristic scale of the modulations. We will consider (2.1) at first as a change of variables from u to the triple ρ, c, w (with $\theta_x = \rho$ and the average of w over y, $\overline{w} = 0$) and find the exact equations that ρ, c and w must satisfy so that u is a solution of KS. Then we solve the equations for ρ, c and w approximately for small ϵ. The transformed KS equation (2.1) has the form

$$\rho^2 L(w) + F^{(\epsilon)}(c, \rho, w) = 0 \tag{2.3}$$

where $F^{(\epsilon)}$ is a function that depends on ρ, c and w and their derivatives (it is given in Appendix A) and L is the linearized operator for (1.2) with ν replaced by $\rho^2\nu$

$$L(\phi) = (U\phi)_y + \phi_{yy} + \rho^2 \nu \phi_{yyyy} \tag{2.4}$$

governing the stability of the cellular solutions $U = U(y, \rho^2\nu)$. As explained further in Appendices A and B, in order to be able to solve (2.3) and have a solution w that averages to zero in the y variable it is necessary that $F^{(\epsilon)}$ have mean zero and that the coefficient of U_y in $F^{(\epsilon)}$ be zero. This leads to two equations for ρ and c which are

$$\rho_t + (c\rho)_x + 3\epsilon\rho_{xx} + \epsilon^3 \nu \left[\frac{(3(\rho\rho_x)_x + \rho\rho_{xx})_x + \rho\rho_{xxx}}{\rho} \right]_x = 0, \tag{2.5}$$

and

$$\begin{aligned} c_t + cc_x + \overline{U^2}\rho\rho_x + 2\nu\overline{UU_\nu}\rho^3\rho_x + \epsilon\overline{U(\rho w)_x} + 2\epsilon\nu\overline{U_\nu w}\rho^2\rho_x \\ + \epsilon c_{xx} + \epsilon^2\overline{ww_x} + \epsilon^3\nu c_{xxxx} = 0, \end{aligned} \tag{2.6}$$

where the overbar stands for average over the y variable. When these equations are satisfied then (2.3) takes the form

$$\begin{aligned} \rho^2 L(w) + F_0(c, \rho; \nu) + \epsilon F_1(c, \rho, w; \nu) + \epsilon^2 F_2(c, \rho, w; \nu) \\ + \epsilon^3 F_3(c, \rho, w; \nu) + \epsilon^4 \nu w_{xxxx} = 0. \end{aligned} \tag{2.7}$$

where F_0, F_1 and F_3 are given in Appendix A.

We will also use below the linearized operator L when $\rho = 1$ and we will then denote it by

$$\mathcal{L}(\phi) = (U\phi)_y + \phi_{yy} + \nu\phi_{yyyy}. \tag{2.8}$$

Some properties of the linearized operator $\mathcal{L}$ (or L) are discussed in Appendix B.

3 Nonlinear modulation theory.

The equations (2.5)-(2.7) are a coupled nonlinear system equivalent to the KS equation. The two modulation equations for ρ and c are, of course, not closed since they depend on the fluctuation term w which in turn depends on ρ and c. To get a modulation theory we must close the ρ and c equations. We do this by expanding the fluctuation w in a power series in ϵ with $\epsilon \to 0$

$$w = w_0 + \epsilon w_1 + \epsilon^2 w_2 + \dots . \tag{3.1}$$

The main point of our modulation analysis is that the modulation equations (2.5) and (2.6) are *exact to all orders* in ϵ and we can make the error be formally smaller than the terms kept if we retain only the three leading order terms in the expansion of w shown above. Substituting (3.1) into (2.5)-(2.7) yields the following equations for the first three terms in the expansion of w:

$$\rho^2 L(w_0) + F_0(c, \rho; \nu) = 0 \tag{3.2a}$$

$$\rho^2 L(w_1) + F_1(c, \rho, w_0; \nu) = 0\,, \tag{3.2b}$$

$$\begin{aligned}
&\rho^2 L(w_2) + F_2(c, \rho, w_0; \nu) + ((\rho w_1)_x U - \overline{(\rho w_1)_x U}) \\
&+2\nu\rho^2 \rho_x (wU_\nu - \overline{wU_\nu}) + w_{1t} + (c w_{1x})_x + \rho(w_0 w_{1y} + w_1 w_{0y}) - 2\rho_x w_{1y} \\
&+2\rho w_{1xy} + 6\nu\rho^2 \rho_x w_{1yyy} + 4\nu\rho^3 w_{1xyyy} \;=\; 0\,.
\end{aligned} \tag{3.2c}$$

The functions F_0, F_1, F_2 are given in Appendix A. Equations (3.2a-c) appear to be linear but they are in fact nonlinear because they are coupled to the modulation equations (2.5) and (2.6). Very little can be said about them at this stage. They must be simplified further. In the next two sections we will study the linearized modulation equations and then their weakly nonlinear form.

4 Linear stability theory.

We now linearize equations (2.5), (2.6) and (3.2a-c) about the exact solution

$$\rho(t,x) = 1 \quad , \quad c(t,x) = 0 \quad , \quad w = 0 \quad .$$

while regarding ϵ as a fixed parameter. We obtain the following linear equations for the small perturbations to ρ, c and w

$$\rho_t + c_x + 3\epsilon\rho_{xx} + 5\epsilon^3 \nu \rho_{xxxx} = 0, \tag{4.1}$$

$$c_t + (\overline{U^2} - 2\nu\overline{UU_\nu})\rho_x + \epsilon\overline{w_x U} + \epsilon c_{xx} + \epsilon^3\nu c_{xxxx} = 0, \tag{4.2}$$

$$\mathcal{L}(w_0) - 2\nu U_\nu c_x + \rho_x(U^2 - \overline{U^2} + 2\nu(UU_\nu - \overline{UU_\nu})$$
$$+4\nu U_{\nu y} + 8\nu^2 U_{\nu yyy} + 10\nu U_{yyy}) = 0, \tag{4.3a}$$

$$\mathcal{L}(w_1) + \rho_{xx}(-2U - 4\nu U_\nu + 10\nu U_{yy} + 12\nu^2 U_{\nu yy})$$
$$+w_{0t} + (w_{0x}U - \overline{w_{0x}U}) + 2w_{0xy} + 4\nu w_{0xyyy} = 0\,, \tag{4.3b}$$

$$\mathcal{L}(w_2) + w_{1t} + w_{0xx} + (w_{1x}U - \overline{w_{1x}U}) + 2w_{1xy}$$
$$+4\nu w_{1xyyy} + 6\nu w_{0xxyy} + 8\nu^2\rho_{xxx}U_{\nu y} = 0\,. \tag{4.3c}$$

The solutions of (4.3a-c) can be written in the form

$$w_0 = \rho_x\chi_1(y) + c_x\chi_2(y),, \tag{4.4}$$

$$w_1 = \rho_{xx}\chi_1^{(1)}(y) + c_{xx}\chi_2^{(1)}(y), \tag{4.5}$$

$$w_2 = \rho_{xxx}\chi_1^{(2)}(y) + c_{xxx}\chi_2^{(2)}(y)\,, \tag{4.6}$$

where the functions χ_1 etc. are functions of y alone and are obtained from

$$\mathcal{L}(\chi_1) + U^2 - \overline{U^2} + 2\nu(UU_\nu - \overline{UU_\nu}) + 4\nu U_{\nu y} + 8\nu^2 U_{\nu yyy} + 10\nu U_{yyy} = 0, \tag{4.7a}$$

$$\mathcal{L}(\chi_2) - 2\nu U_\nu = 0, \tag{4.7b}$$

$$\mathcal{L}(\chi_1^{(1)}) + (\chi_1 U - \overline{\chi_1 U}) - (\overline{U^2} + 2\nu\overline{UU_\nu})\chi_2 + 2\chi_{1y}$$
$$+4\nu\chi_{1yyy} - 2U - 4\nu U_\nu + 10\nu U_{yy} + 12\nu^2 U_{\nu yy} = 0\,, \tag{4.7c}$$

$$\mathcal{L}(\chi_1^{(2)}) + (\chi_2 U - \overline{\chi_2 U}) - \chi_1 + 2\chi_{2y} + 4\nu\chi_{2yyy} = 0\,, \tag{4.7d}$$

$$\mathcal{L}(\chi_2^{(1)}) + (\chi_1^{(1)}U - \overline{\chi_1^{(1)}}) - (\overline{U^2} + 2\nu\overline{UU_\nu})\chi_2^{(1)} + 2\chi_{1y}^{(1)} + 4\nu\chi_{1yyy}^{(1)}$$

$$6\nu\chi_{1yy} - 2\chi_1 - (\overline{U\chi_1})\chi_2 + 8\nu^2 U_{\nu y} = 0\,, \tag{4.7e}$$

$$\mathcal{L}(\chi_2^{(2)}) + (\chi_2^{(1)}U - \overline{\chi_2^{(1)}U}) - \chi_1^{(1)} + 2\chi_{2y}^{(1)}$$
$$+4\nu\chi_{2yyy}^{(1)} + 6\nu\chi_{2yy} - (\overline{U\chi_2})\chi_2 = 0\,. \tag{4.7f}$$

It is convenient to define the weighted averages of the χ's with weight the cellular solution $U(y)$. We let

$$\alpha_1 = \overline{U\chi_1}\ ,\ \alpha_2 = \overline{U\chi_2}\ ,\ \alpha_3 = \overline{U\chi_1^{(1)}}\ ,$$
$$\alpha_4 = \overline{U\chi_2^{(1)}}\ ,\ \alpha_5 = \overline{U\chi_1^{(2)}}\ ,\ \alpha_6 = \overline{U\chi_2^{(2)}}\,. \tag{4.8}$$

Using (4.4)-(4.6) and the definitions (4.8) in the linearized modulation equations (4.1) and (4.2) gives

$$\rho_t + c_x + 3\epsilon\rho_{xx} + 5\epsilon^3\nu\rho_{xxxx} = 0, \tag{4.9}$$

$$c_t + (U^2 + 2\nu\overline{UU_\nu})\rho_x + \epsilon\alpha_1\rho_{xx} + \epsilon(1+\alpha_2)c_{xx}$$
$$+\epsilon^2(\alpha_3\rho_{xxx} + \alpha_4 c_{xxx}) + \epsilon^3\alpha_5\rho_{xxxx} + \epsilon^3(\alpha_6+\nu)c_{xxxx} = 0. \tag{4.10}$$

Equations (4.9) and (4.10) contain the linear modulational or long-wave stability characteristics of the cellular steady states U of the Kuramoto-Sivashinsky equation up to order ϵ^3. The parameters of the problem are the coefficients $\alpha_1, ..., \alpha_6$ which are determined numerically as described in Section 6.

To analyze the stability of U it is sufficient to look for plane-wave solutions of the form

$$(\rho(t,x), c(t,x)) = (\overline{\rho}, \overline{c})e^{ikx+\omega t}\quad , \tag{4.11}$$

where $\overline{\rho}, \overline{c}$ are constants, k is the wavenumber of a particular disturbance and ω is the corresponding growth rate. Substituting (4.11) into (4.9) and (4.10) gives the dispersion relation that determines ω as a function of k

$$\omega^2 + \omega\left[-\epsilon(4+\alpha_2)k^2 + \epsilon^3(6\nu+\alpha_6)k^4\right] + k^2\lambda^2$$
$$+\epsilon^2 k^4(3+3\alpha_2-\alpha_3) - \epsilon^4 k^6(8\nu+3\alpha_6+5\alpha_2\nu) + 5\epsilon^6 k^8\nu(\nu+\alpha_6) \ = \ 0\ . \tag{4.12}$$

Here

$$\lambda^2 = \overline{U^2} + 2\nu\overline{UU_\nu}\ \ .$$

and in deriving this relation we have anticipated (6.6). We note here that λ^2 can be either positive or negative, a fact that is central to the modulational stability of the cellular solution. It is easy to get the exact solution of the quadratic equation for ω. The main feature of our

analysis here is the inclusion of higher order terms in ϵ. We will focus on the (linearized) structural stability of the celular solution of KS. For k large we have

$$\omega = \pm ik\lambda + \epsilon\omega_1 k^2 + \dots \quad ,$$

where

$$\omega_1 = \frac{1}{2}(4 + \alpha_2) \quad . \tag{4.13}$$

The parameter ω_1 plays the role of a diffusion coefficient. Clearly the system is stable if both $\lambda^2 > 0$ and $\omega_1 < 0$. Such a two-term result obtained in [1] in their related study. Based on the two-term result, they determined a range of values of ν for which the cellular solutions are linearly stable. It turns out, however, that the extent of this region is very small and consequently of limited significance (see [1] and our Section 7). The two-term analysis gives an incomplete answer to the linear and weakly nonlinear stability of the system because it does not account for the effects of the *stabilizing* fourth derivative terms in the modulation equations. These contributions are easily computed here. We find that there is a correction term of $O(i\epsilon^2 k^3)$ to ω which is dispersive and does not contribute to growth or decay. The physically significant for stability correction is obtainable from a large k analysis of the dispersion relation (4.13), and is given by

$$\omega = \epsilon^3 k^4 \omega_3 \quad , \quad \omega_3 = -\nu - \alpha_6 \;\; or \;\; -5\nu \quad . \tag{4.14}$$

The results (4.13), (4.14) can be used to determine the structural stability of the cellular solutions. First, a range of ν must be found which satisfies $\lambda^2 > 0$ so that stable solutions are obtained to leading order. The next correction to the real part of ω is $\epsilon k^2 \omega_1$. For stability this term must be negative, and so the range of ν for which $\lambda^2 > 0$ and $-\omega_1 > 0$ is a linear stability region. As mentioned earlier, this range of ν is very small because the next correction to the real part of ω given by (4.14) is neglected. We have structural stability when ν is such that $-\omega_3 = \nu + \alpha_6 > 0$. When ω_1 is positive but ω_3 is negative then we have structural stability and a finite band of linearly unstable wave numbers. This is a regime of ν values for which the weakly nonlinear modulation theory of the next section is applicable, and it is much larger than the one for which ω_1 is negative [1]. The computations leading to these conclusions are described in Section 7.

5 Weakly nonlinear theory.

To get a suitable weakly nonlinear modulation theory we must return to (2.5), (2.6) and (3.2a-c) and simplify them in a way consistent with the expansion (3.1) without, however, altering the construction of the χ functions of the linearized theory. This can be achieved by

making the size of the perturbations in ρ, c and w about 1, 0 and 0, respectively, of order ϵ^3, in which case nonlinear terms enter into the modulation equations for c and ρ and are comparable with the stabilizing fourth derivative terms. We therefore let

$$\rho(t,x) = 1 + \epsilon^3\tilde{\rho}(t,x)\ ,\ \ c(t,x) = \epsilon^3\tilde{c}(t,x)\ ,\ \ w(t,x) = \epsilon^3\tilde{w}(t,x) \tag{5.1}$$

and substitute this into (2.4)-(2.6) to get the following equations for ρ and c (with the tildes omitted)

$$\rho_t + c_x + 3\epsilon\rho_{xx} + \epsilon^3(c\rho)_x + 5\epsilon^3\nu\rho_{xxxx} = 0 \quad , \tag{5.2}$$

$$\begin{aligned} &c_t + \lambda^2\rho_x + \epsilon(1+\alpha_2)c_{xx} + \epsilon^2\alpha_3\rho_{xxx} \\ &+\epsilon^3(cc_x + \lambda^2\rho\rho_x) + \epsilon^3(\alpha_6+\nu)c_{xxxx} = 0 \quad . \end{aligned} \tag{5.3}$$

These are the weakly nonlinear modulation equations. The difference between the system (5.2) and (5.3) and the linear system (4.10) and (4.11) is the appearance of the nonlinear terms of $O(\epsilon^3)$ in the former. The constants λ^2, α_2, α_3, α_6 for (5.2) and (5.3) are identical to the constants of the linear theory in Section 4.

6 Computation of stability parameters.

We must solve problems of the form

$$\mathcal{L}(\chi) = f(y) \quad , \quad \chi(y+2\pi) = \chi(y) \quad , \quad \overline{\chi} = 0 \quad , \tag{6.1}$$

with $f(y)$ a known 2π-periodic function of mean zero and the operator $\mathcal{L}$ defined by (2.8). Since $\chi(y)$ and $f(y)$ are periodic in y and have mean zero, we can find periodic functions $\psi(y)$ and $F(y)$ such that

$$\chi = \psi_y \quad , \quad f = F_y \ . \tag{6.2}$$

Substituting into (6.1) and integrating with respect to y gives

$$U\psi_y - \overline{U\psi_y} + \psi_{yy} + \nu\psi_{yyyy} = F \ , \quad \overline{F} = 0, \tag{6.3}$$

with ψ determined only up to a constant which we take to be zero. It is convenient to introduce the operator $\mathcal{M}$ defined by

$$\mathcal{M}(\psi) = U\psi_y - \overline{U\psi_y} + \psi_{yy} + \nu\psi_{yyyy} \quad .$$

Integrating each of the equations (4.7a-f) gives the following sequence of problems

$$\mathcal{M}(\psi_1) + \int^y (U^2 - \overline{U^2})dy + 2\nu\int^y (UU_\nu - \overline{UU_\nu})dy$$

$$+10\nu U_{yy} + 8\nu^2 U_{\nu yy} + 4\nu U_\nu = 0\,, \tag{6.3a}$$

$$\mathcal{M}(\psi_2) - 2\nu \int^y U_\nu dy = 0\,, \tag{6.3b}$$

$$\begin{aligned} \mathcal{M}(\psi_1^{(1)}) + \int^y (\psi_{1y} U - \overline{\psi_{1y} U}) dy - (\overline{U^2} + 2\nu \overline{U U_\nu})\psi_2 + 2\psi_{1y} + 4\nu\psi_{1yyy} \\ -2\int^y U dy - 4\nu \int^y U_\nu dy + 10\nu U_y + 12\nu^2 U_{\nu y} = 0\,, \end{aligned} \tag{6.3c}$$

$$\mathcal{M}(\psi_2^{(1)}) + \int^y (\psi_{2y} U - \overline{\psi_{2y} U}) dy - \psi_1 + 2\psi_{2y} + 4\nu\psi_{2yyy} = 0\,, \tag{6.3d}$$

$$\begin{aligned} \mathcal{M}(\psi_1^{(2)}) + \int^y (\psi_{1y}^{(1)} U - \overline{\psi_1^{(1)}} U) dy - (\overline{U^2} + 2\nu \overline{U U_\nu})\psi_2^{(1)} + 2\psi_{1y}^{(1)} \\ +4\nu\psi_{1yyy}^{(1)} + 6\nu\psi_{1yy} + 8\nu^2 U_\nu - 2\psi_1 = 0\,, \end{aligned} \tag{6.3e}$$

$$\begin{aligned} \mathcal{M}(\psi_2^{(2)}) + \int^y (\psi_{2y}^{(1)} U - \overline{\psi_{2y}^{(1)} U}) dy - \psi_1^{(1)} + 2\psi_{2y}^{(1)} \\ +4\nu\psi_{2yyy}^{(1)} + 6\nu\psi_{2yy} - (\overline{U\psi_{2y}})\psi_2 = 0\,. \end{aligned} \tag{6.3f}$$

The function U_ν is needed in the calculations and is computed by solving

$$\mathcal{M}(V) = -U_{yyyy} \quad , \quad U_\nu = V_y \quad . \tag{6.3g}$$

We now solve numerically the integrated equations (6.3a-f). We have also solved (5.7a-f) by a similar method and with almost identical results but (6.3a-f) is more convenient. The central element in the construction of numerical solutions of (6.3a-f) (or (6.1)) is the steady state, periodic solutions U of the Kuramoto-Sivashinsky equation, which can be described by a very small number of Fourier modes. Since we are interested in a range of values of ν (approximately $0.3 < \nu < 1.0$) which is above any bifurcations, we know that steady states are unique global attractors. The numerical calculations of [34, 35] show that a practical estimate of the number of modes required is a few more than $\nu^{-1/2}$, the number of linearly unstable ones. We therefore write

$$U(y;\nu) = \sum_{n=1}^{N} U_n \sin(ny) \quad ,$$

where N and U_n depend on ν and are found numerically. With this approximate solution, the linear operator (6.3) becomes a matrix equation for the constants a_n, b_n where

$$\psi(y) = \sum_{n=1}^{N}(a_n \sin(ny) + b_n \cos(ny)). \tag{6.4}$$

Expanding also the forcing $F(y)$

$$F(y) = \sum_{n=1}^{N}(f_n \sin(ny) + g_n \cos(ny)),$$

the system of linear equations to be solved is

$$\sum_{j=1}^{m-1} \frac{1}{2} j a_j U_{m-j} + \sum_{j=1}^{N-m} \frac{1}{2}(j a_j U_{j+m} - (j+m) a_{j+m} U_j)$$
$$+m^2(\nu m^2 - 1)a_m = f_m \quad , \tag{6.5a}$$

$$\sum_{j=1}^{m-1} \frac{1}{2} j b_j U_{m-j} - \sum_{j=1}^{N-m} \frac{1}{2}(j b_j U_{j+m} + (j+m) b_{j+m} U_j)$$
$$+m^2(\nu m^2 - 1)b_m = g_m \quad , \tag{6.5b}$$

for $1 \leq m \leq N$ where the first or second sums are omitted when $m = 1$ or N respectively. We see from (6.5a,b) that the equations for the sine and cosine coefficients of ψ are decoupled. In particular if $f_m = 0$ ($g_m = 0$), for $m = 1, N$ then $a_m = 0$ ($b_m = 0$), $m = 1, N$ also. This holds as long as the determinant of the $n \times n$ matrix defined by (6.5a) or (6.5b) respectively, is non-vanishing. This condition is monitored throughout the computation and is also checked analytically near the bifurcation point $\nu = 1$ (see Appendix C). The symmetry is due to the odd parity of the steady state $U(y;\nu)$. We expect, therefore, that

$$\alpha_1 = 0 \quad , \quad \alpha_4 = 0 \quad , \quad \alpha_5 = 0 \quad . \tag{6.6}$$

For a given value of ν the coefficients U_n were obtained numerically by solving the unsteady KS equation until a steady state is reached (see [35] for a description of the numerical scheme). For values of ν near 1, the integration requires longer times to reach a steady state. The results reported here have the same accuracy for all values of ν.

With the Fourier coefficients of $U(y;\nu)$ known, the solution vector $(a_1, a_2, ..., a_N, b_1, ..., b_N)^t$ is found by a single matrix inversion for each value of ν. The stability constants $\alpha_i, i = 1, ..., 6$ are determined easily from their definitions (5.8). The symmetry result (6.6) is confirmed by our computations which do not explicitly assume the odd-parity property of $U(y)$ and can therefore be extended to general steady states. We note, however, that for the range of ν studied here, any steady state can be described by the odd-parity solutions by means of a suitable translation or Galilean transformation, if necessary.

7 Numerical results for linear stability.

We now present the results of our computations of the stability constants $\alpha_1, ..., \alpha_6$. First we consider the behavior of the solutions near the bifurcation point $\nu = 1$. This analysis is given in Appendix C and the main results are that

$$\lambda^2 = -24 + O(\xi) \quad , \quad \alpha_2 = -\frac{12}{\xi} + 28 - 16\xi \tag{7.1}$$

where $\xi = 1 - \nu$. Analogous results for the asymptotic form of ψ_2 are given in Appendix C. Table 1 below compares the asymptotic results for α_2 with direct computations when $N = 10$.

ξ ($\nu = 1 - \xi$)	Exact (N=10)	Asymptotic
0.11	-84.61	-82.85
0.12	-72.62	-73.92
0.13	-62.84	-66.39
0.14	-54.73	-59.95
0.15	-47.90	-54.40

The asymptotic results show that for small ξ the solutions are unstable to leading order. The next order correction, however, is stable since $\omega_1 < 0$ for small ξ. For general values of ν the coefficients are computed numerically as explained in Section 6. In Table 2 below we present the results for the parameters λ^2 and ω_1 for a range of values of ν. The table also gives the values of $\overline{U}$. These results were produced with $N = 6$ or 10 and no change in the accuracy was observed.

ν	λ^2	$-\omega_1$	$\overline{U}$
0.89	-14.59	40.30	2.11
0.85	-10.88	21.95	2.79
0.80	-6.84	10.77	3.51
0.75	-3.33	4.73	4.09
0.70	-0.287	1.07	4.51
0.65	2.30	-1.29	4.79
0.60	4.43	-2.86	4.93
0.55	6.14	-3.94	4.92
0.50	7.41	-4.69	4.74
0.45	8.23	-5.22	4.41
0.40	8.70	-5.64	3.92
0.35	8.77	-6.09	3.25
0.30	8.57	-6.97	2.37

From Table 2 we see that there is a small region, approximately $0.65 < \nu < 0.7$, where both λ^2 and $-\omega_1$ are positive and so the solutions are stable. The results of Table 2 are shown in Figure 1. The stability window (the same as in [1]) is present but is very small.

When $\omega_3 < 0$ then there is a stabilization by the fourth derivative and the structural stability window is enlarged to $0.36 < \nu < 0.7$, approximately. Note that $-\omega_3$ becomes negative somewhere in $0.36 < \nu < 0.365$ and in $0.695 < \nu < 0.7$, respectively.

8 Numerical solution of the weakly nonlinear modulation equations

We will solve numerically equations (5.2) and (5.3). We consider the following initial value problem

$$\rho_t + c_x + 3\epsilon\rho_{xx} + \epsilon^3(\rho c)_x + 5\epsilon^3\nu\rho_{xxxx} = 0 \quad , \tag{8.1}$$

$$\begin{aligned} &c_t + \lambda^2\rho_x + \epsilon(1+\alpha_2)c_x x + \epsilon^2\alpha_3\rho_{xxx} \\ &+\epsilon^3(cc_x + \lambda^2\rho\rho_x) + \epsilon^3(\alpha_6+\nu)c_{xxxx} = 0\,, \end{aligned} \tag{8.2}$$

$$\rho(x,0) = \rho_0(x) \quad , \quad c(x,0) = c_0(x)\,, \tag{IC}$$

where the initial conditions satisfy

$$\rho \to 1 \quad , \quad c \to 0 \quad as \quad |x| \to \infty\,. \tag{BC}$$

We want to see how a localized perturbation in ρ and c will evolve, when ν and ϵ take different values.

For the numerical work it is convenient to transform to new variables $\rho \to 1 + A\rho$ and $c \to Ac$ which makes the boundary conditions homogeneous, so that (8.1) and (8.2) become

$$\rho_t + (1+\epsilon^3)c_x + 3\epsilon\rho_{xx} + \epsilon^3 A(\rho c)_x + 5\epsilon^3\nu\rho_{xxxx} = 0 \quad , \tag{8.3}$$

$$\begin{aligned} &c_t + \lambda^2(1+\epsilon^3)\rho_x + \epsilon(1+\alpha_2)c_x x + \epsilon^2\alpha_3\rho_{xxx} \\ &+\epsilon^3 A(cc_x + \lambda^2\rho\rho_x) + \epsilon^3(\alpha_6+\nu)c_{xxxx} = 0 \quad . \end{aligned} \tag{8.4}$$

The constant A is a measure of the nonlinearity and in the numerical experiments it is such that $\epsilon^3 A = 0.1$ or 0.5.

We will use a spectral method to represent spatial derivatives and a time splitting scheme. We will therefore be solving (8.3) and (8.4) on a large (compared to the support of the initial data) but finite spatial region, with periodic boundary conditions. We will integrate the equations up to the time that ρ and c disturbances reach the ends of the spatial region, since unphysical effects due to the boundary conditions will then be felt which are inconsistent

with the modulation theory. For the time splitting scheme, which is second order accurate, the nonlinear and linear terms are treated separately. The split equations are

$$\rho_t + \epsilon^3 A(\rho c)_x = 0\,, \tag{8.5}$$

$$c_t + \frac{1}{2}\epsilon^3 A(c^2 + \lambda^2\rho^2)_x = 0\,, \tag{8.6}$$

and

$$\rho_t + (1+\epsilon^3)c_x + 3\epsilon\rho_{xx} + 5\epsilon^3\nu\rho_{xxxx} = 0 \quad , \tag{8.7}$$

$$c_t + \lambda^2(1+\epsilon^3)\rho_x + \epsilon(1+\alpha_2)c_{xx} + \epsilon^2\alpha_3\rho_{xxx} + \epsilon^3(\alpha_6+\nu)c_{xxxx} = 0 \quad . \tag{8.8}$$

The nonlinear equations (8.5) and (8.6) are a pair of conservation laws. It is easy to see that the system is strictly hyperbolic with eigenvalues

$$\mu = \epsilon^3 A(c \pm \lambda\rho)\ ,$$

the strict hyperbolicity being a consequence of $\lambda^2 > 0$ for all ν of interest. The time integration is done by a second order accurate Lax-Wendroff scheme with no artificial viscosity. The linear part of the equations is solved exactly in Fourier space. Denoting Fourier transforms by a caret, the solutions of (8.7), (8.8) subject to initial conditions (IC) are given by

$$\hat{\rho}(k,t) = A_1 e^{\omega_1(k)t} + A_2 e^{\omega_2(k)t} \quad , \tag{8.9}$$

$$\hat{c}(k,t) = B_1 e^{\omega_1(k)t} + B_2 e^{\omega_2(k)t} \quad , \tag{8.10}$$

where A_1, A_2, B_1, B_2 are constants determined from the initial conditions

$$\hat{\rho}(k,0) = \hat{\rho}_0(k) \quad , \quad \hat{c}(k,0) = \hat{c}_0(k) \quad , \tag{8.11}$$

$$\hat{\rho}_t(k,0) = (3\epsilon k^2 - 5\epsilon^3\nu k^4)\hat{\rho}_0(k) - ik(1+\epsilon^3)\hat{c}_0(k) \quad , \tag{8.12}$$

$$\hat{c}_t(k,0) = (i\epsilon^2k^3\alpha_3 - ik\lambda^2(1+\epsilon^3))\hat{\rho}_0(k) + (\epsilon k^2(1+\alpha_2) - \epsilon^3k^4(\nu+\alpha_6))\hat{c}_0(k) \quad , \tag{8.13}$$

and given by

$$A_1 = \frac{\hat{\rho}_t - \omega_2\hat{\rho}_0}{\omega_1 - \omega_2} \quad , \quad A_2 = \frac{\hat{\rho}_t - \omega_1\hat{\rho}_0}{\omega_2 - \omega_1}\,,$$

$$B_1 = \frac{\hat{c}_t - \omega_2\hat{c}_0}{\omega_1 - \omega_2} \quad , \quad B_2 = \frac{\hat{c}_t - \omega_1\hat{c}_0}{\omega_2 - \omega_1} \quad .$$

The eigenvalues ω_1, ω_2 are the two roots of the dispersion relation

$$\omega^2 + \omega\left[-\epsilon(4+\alpha_2)k^2 + \epsilon^3k^4(6\nu+\alpha_6)\right] + (1+\epsilon^3)^2k^2\lambda^2$$

$$+\epsilon^2k^4(3+3\alpha_2-(1+\epsilon^3)\alpha_3) - \epsilon^4k^6(8\nu+3\alpha_6+5\nu\alpha_2) + 5\epsilon^6k^8\nu(\nu+\alpha_6) = 0 \quad . \tag{8.14}$$

The solution of the linear part of the problem can then be advanced to the level $t+\Delta t$ once the spectrum of ρ and c are known at level t. A further accuracy requirement, particularly in the case of imaginary values of ω_1 and ω_2, is that the exponents $\omega_1\Delta t$ and $\omega_2\Delta t$ must be smaller in absolute value than a certain tolerance, typically no larger than 0.1 in our computations.

Representative computations are given for the values $\nu = 0.69$ and $\nu = 0.55$. The corresponding values of ϵ used are 0.008 and 0.01 respectively. The linear spectra for these parameters computed from the dispersion relation (8.14) are given in Figures 2, 3 and 4 and the nonlinear evolutions in Figures 5, 6a-b and 7a-b.

9 Conclusions

We have considered the modulational stability of a class of steady spatially periodic solutions of the Kuramoto-Sivashinsky equation. The theory is initially developed in a fully nonlinear framework and analytical and computational results are presented for linear and weakly nonlinear regimes. We find that the periodic waves are much more stable than previously thought due to a high-order stabilizing correction which follows easily from our theory. Nonlinear evolution solutions are also presented and indicate spatial spreading of the disturbances.

Appendices

A Derivation of the exact nonlinear stability equations

In this Appendix we derive equations (2.5), (2.6) and (3.2a-c). Substituting the exact transformation (2.1) into the KS equation (2.2) and noting that

$$\frac{\partial}{\partial t} \to \frac{\partial}{\partial t} + \frac{1}{\epsilon}\theta_t\frac{\partial}{\partial y} + 2\nu\rho\rho_t\frac{\partial}{\partial \nu},$$

$$\frac{\partial}{\partial x} \to \frac{\partial}{\partial t} + \frac{1}{\epsilon}\rho\frac{\partial}{\partial y} + 2\nu\rho\rho_x\frac{\partial}{\partial \nu},$$

with the $\frac{\partial}{\partial \nu}$ derivative acting only on U, yields the transformed KS equation in which ϵ is an order parameter.

$$\frac{1}{\epsilon^3}\Big[\Big(\rho\theta_t + \rho^2 c + 3\epsilon\rho\rho_x + \epsilon^3\nu(\rho\rho_{xxx} + F_x)\Big)\, U_y$$

$$
\begin{aligned}
&+\rho^3(UU_y+U_{yy}+\rho^2\nu U_{yyyy})\Big]+\frac{1}{\epsilon^2}\Big[\rho^2((Uw)_y+w_{yy}+\rho^2\nu w_{yyyy})\\
&+(\rho_t+(c\rho)_x)U+(\theta_t+c\rho)w_y+2\nu\rho^2(\rho_t+c\rho_x)U_\nu+4\nu\rho^3\rho_xU_{\nu y}+8\nu^2\rho^5\rho_xU_{\nu yyy}\\
&+\rho\rho_xU^2+2\nu\rho^3\rho_xUU_\nu+10\nu\rho^3\rho_xU_{yyy}+c_t+cc_x+\epsilon(\rho w)_xU\\
&+2\epsilon\nu\rho^2\rho_xU_\nu w+\epsilon c_{xx}+\epsilon^2ww_x+\epsilon^3\nu c_{xxxx}\Big]\\
&+\frac{1}{\epsilon}\Big[w_t+(cw)_x+\rho ww_y+\rho_{xx}U+2\nu((\rho^2\rho_x)_x+\rho\rho_x^2)U_\nu+4\nu^2\rho^3\rho_x^2U_{\nu\nu}\\
&+\rho_xw_y+2\rho w_{xy}+\nu(\rho F+6(\rho^2\rho_x)_x)U_{yy}+2\nu^2(\rho F^{\nu y}+3(\rho^4\rho_x)_x+6\rho^3\rho_x^2)U_{\nu yy}\\
&+24\nu^3\rho^5\rho_x^2U_{\nu\nu yy}+6\nu\rho^2\rho_xw_{yyy}+4\nu\rho^3w_{xyyy}\Big]\\
&\Big[w_{xx}+2\nu^2(\rho F^\nu+F_x^{\nu y}+\rho\rho_xF)U_{\nu y}+4\nu^3(\rho F^{\nu\nu}+3(\rho^4\rho_x^2)_x+\rho\rho_xF^{\nu y})U_{\nu\nu y}\\
&+32\nu^4\rho^5\rho_x^3U_{\nu\nu\nu y}+12\nu\rho\rho_xw_{xyy}+6\nu\rho^2w_{xxyy}+\nu Fw_{yy}\Big]\\
&+\epsilon\nu\Big[\rho_{xxxx}U+2\nu(F_x^\nu+\rho\rho_x\rho_{xxx})U_\nu+4\nu^2(F_x^{\nu\nu}+\rho\rho_xF^\nu)U_{\nu\nu}\\
&+8\nu^3((\rho^4\rho_x^3)_x+\rho\rho_xF^{\nu\nu})U_{\nu\nu\nu}+16\nu^4\rho^5\rho_x^4U_{\nu\nu\nu\nu}+4\rho w_{xxxy}\\
&+4\rho_{xx}w_{xy}+6\rho_xw_{xxy}+\rho_{xxx}w_y\Big]+\epsilon^2\nu w_{xxxx}=0.
\end{aligned}
\tag{A1}
$$

Here

$$
\begin{aligned}
F&=3(\rho\rho_x)_x+\rho\rho_{xx}\,,\\
F^\nu&=(\rho^2\rho_x)_{xx}+(\rho\rho_x^2)_x+\rho\rho_x\rho_{xx}\,,\\
F^{\nu y}&=2(\rho^3\rho_x)_x+4\rho^2\rho_x^2+\rho(\rho^2\rho_x)_x\,,\\
F^{\nu\nu}&=(\rho^3\rho_x^2)_x+\rho\rho_x(\rho^2\rho_x)_x+\rho^2\rho_x^3\,.
\end{aligned}
$$

Equation (A1) above has been rearranged in ascending powers of ϵ and certain terms have been grouped together as we explain next. The objective is to derive exact nonlinear modulational equations which hold in the asymptotic regime $\epsilon \ll 1$ *without* the need of cumbersome solvability conditions at each order of the expansion. The modulation equations act as all order solvability conditions. This is achieved by making the fluctuation w^ϵ have zero mean. Since the determination of w^ϵ depends on solutions of an equation of the form $L(w^\epsilon)=f(y)$ where $f(y)$ is a known forcing function, it is necessary that f have mean zero and that w^ϵ has zero mean if $f(y)$ does not contain terms proportional to U_y. All terms proportional to U_y, therefore, are grouped together, as shown in (A1) above, and c, ρ are chosen so that the mean of f and the coefficient of U_y vanishes for all x and t. The equations are

$$
\rho_t+(c\rho)_x+3\epsilon\rho_{xx}+\epsilon^3\nu\left[\frac{(3(\rho\rho_x)_x+\rho\rho_{xx})_x+\rho\rho_{xxx}}{\rho}\right]_x=0, \tag{A2}
$$

$$c_t + cc_x + \overline{U^2}\rho\rho_x + 2\nu\overline{UU_\nu}\rho^3\rho_x + \epsilon\overline{U(\rho w)_x} + 2\epsilon\nu\overline{U_\nu w}\rho^2\rho_x$$
$$+\epsilon c_{xx} + \epsilon^2\overline{ww_x} + \epsilon^3\nu c_{xxxx} = 0\,. \qquad (A3)$$

Note that (2.3a) is obtained by dividing (A2) by ρ and differenting with respect to x. With the above choice of modulation functions, the equation that governs the fluctuation w becomes

$$\rho^2 L(w) + F_0 + \epsilon F_1 + \epsilon^2 F_2 + \epsilon^3\nu F_3 + \epsilon^4\nu w_{xxxx} = 0.$$

where the functions F_0, F_1, F_2, F_3 are

$$F_0 = -2\nu\rho^3 c_x U_\nu + 10\nu\rho^3\rho_x U_{yyy} + 8\nu^2\rho^5\rho_x U_{\nu yyy}$$
$$+4\nu\rho^3\rho_x U_{\nu y} + \rho\rho_x(U^2 - \overline{U^2}) + 2\nu\rho^3\rho_x(UU_\nu - \overline{UU_\nu})\,,$$

$$F_1 = (\rho w)_x U - \overline{(\rho w)_x U} + 2\nu\rho^2\rho_x(wU_\nu - \overline{wU_\nu}) + w_t + (cw)_x$$
$$+\rho w w_y - 2\rho_x w_y - 6\nu\rho^2\rho_{xx}U_\nu - 3\rho_{xx} + \rho_{xx}U + 2\nu(\rho\rho_x^2 + (\rho^2\rho_x)_x)U_\nu$$
$$+4\nu^2\rho^3\rho_x^2 U_{\nu\nu} + \nu(\rho F + 6(\rho^2\rho_x)_x)U_{yy} + 2\nu^2(\rho F^{\nu y} + 3(\rho^4\rho_x)_x + 6\rho^3\rho_x^2)U_{\nu yy}$$
$$+24\nu^3\rho^5\rho_x^3 U_{\nu\nu yy} + 2\rho w_{xy} + 6\nu\rho^2\rho_x w_{yyy} + 4\nu\rho^3 w_{xyyy}\,,$$

$$F_2 = 2\nu^2(\rho F^\nu + F_x^{\nu y} + \rho\rho_x F)U_{\nu y} + 4\nu^3(\rho F^{\nu\nu} + 3(\rho^4\rho_x^2)_x + \rho\rho_x F^{\nu y})U_{\nu\nu y}$$
$$+32\nu^4\rho^5\rho_x^3 U_{\nu\nu\nu y} + w_{xx} + ww_x - \overline{ww_x} + 12\nu\rho\rho_x w_{xyy} + 6\nu\rho^2 w_{xxyy} + \nu(\rho\rho_{xx} + 3(\rho\rho_x)_x)w_{yy}\,,$$

$$F_3 = \left[\rho_{xxxx} - \left(\frac{\rho\rho_{xxx} + F_x}{\rho}\right)_x\right] U + 2\nu\left[F_x^\nu + \rho\rho_x\rho_{xxx} - \rho^2\left(\frac{\rho\rho_{xxx} + F_x}{\rho}\right)_x\right] U_\nu$$
$$+4\nu^2(F_x^{\nu\nu} + \rho\rho_x F^\nu)U_{\nu\nu} + 8\nu^3((\rho^4\rho_x^3)_x + \rho\rho_x F^{\nu\nu})U_{\nu\nu\nu} + 16\nu^4\rho^5\rho_x^4 U_{\nu\nu\nu\nu}$$
$$+\left[\rho_{xxx} - \frac{\rho\rho_{xxx} + F_x}{\rho}\right] w_y + 4\rho w_{xxxy} + 4\rho_{xx}w_{xy} + 6\rho_x w_{xxy}\,,$$

B Properties of the linear operator $\mathcal{L}$.

At several points in the analysis we have to solve

$$\mathcal{L}(w) = h, \qquad (B1)$$

where, omitting x and t dependence, $h \equiv h(y)$ is a given forcing function with mean zero and $w = w(y)$ is also a periodic function with mean zero. The operator $\mathcal{L}$ is defined by (2.8). It is clear that

$$\mathcal{L}(1) = U_y$$

and so to make the solution of (B1) have mean zero we must set to zero the coefficient of U_y in h. Thus, the solvability condition for (B1) is

- $\overline{h} = 0$, and
- The coefficient of U_y in h (if any) is zero.

C Stability analysis near $\nu = 1$.

We are interested in the modulational stability of the cellular solution U for values of ν just below 1. Let $\nu = 1 - \xi$ where $\xi \ll 1$. A two-term Fourier expansion is sufficient to describe the steady state $U(y)$:

$$U(y;\xi) = U_1 \sin(y) + U_2 \sin(2y) + \dots \quad ,$$

$$U_1 = -2\ 12^{1/2}\xi^{1/2} \quad , \quad U_2 = -2\xi \quad . \tag{C1}$$

We also need U_ν which from (C1) is

$$U_\nu(y;\xi) = 12^{1/2}\xi^{-1/2}\sin(y) + 2\sin(2y) + \dots \quad . \tag{C2}$$

To fix matters, consider the asymptotic solutions for ψ_2 from equation (6.3b). This solution leads directly to the value of α_2 which is one of the important stability parameters. The solution ψ_2 is

$$\psi_2(y) = a_1 \sin(y) + a_2 \sin(2y) + b_1 \cos(y) + b_2 \cos(2y) + \dots \quad . \tag{C3}$$

Since the forcing terms of (6.3b) can be expanded in a series of cosines, we expect $a_1 = a_2 = 0$. This holds for general values of ν (verified numerically), and it is shown here that the result also holds near $\nu = 1$. Substituting (C3) into (6.5a) and (6.5b) with $N = 2$ and $m = 1, 2$ yields the decoupled linear equations

$$(\frac{1}{2}U_2 + (\nu - 1))a_1 - U_1 a_2 = 0 \quad , \tag{C4.1}$$

$$\frac{1}{2}U_1 a_1 + 4(4\nu - 1)a_2 = 0 \quad , \tag{C4.2}$$

$$(-\frac{1}{2}U_2 + (\nu - 1))b_1 - U_1 b_2 = -2(1-\xi)12^{1/2}\xi^{-1/2} \quad , \tag{C5.1}$$

$$\frac{1}{2}U_1 b_1 + 4(4\nu - 1)b_2 = -2(1-\xi) \quad . \tag{C5.2}$$

The determinant of the coefficient matrix of the first system is $D = 32\xi^2$, and so the trivial solutions $a_1 = a_2 = 0$ follow as expected. The second system gives

$$b_1 = -\frac{1}{12^{1/2}}(\frac{12}{\xi^{3/2}} - \frac{30}{\xi^{1/2}} + 18\xi^{1/2}) \quad , \quad b_2 = -\frac{1}{\xi} + 1 \quad . \tag{C6}$$

These expressions are exact in ξ. From the definition (4.9) of α_2 and using $\chi_2 = \psi_{2y}$ we find

$$\alpha_2 = -\frac{1}{2}U_1 b_1 - U_2 b_2 = -\frac{12}{\xi} + 28 - 16\xi \quad . \tag{C7}$$

We note that this result can also be obtained from an asymptotic analysis of the differentiated equation (5.7b) for χ_2. Comparison of the asymptotic expression (C6) with the direct simulations shows that they are in excellent agreement (see Section 7).

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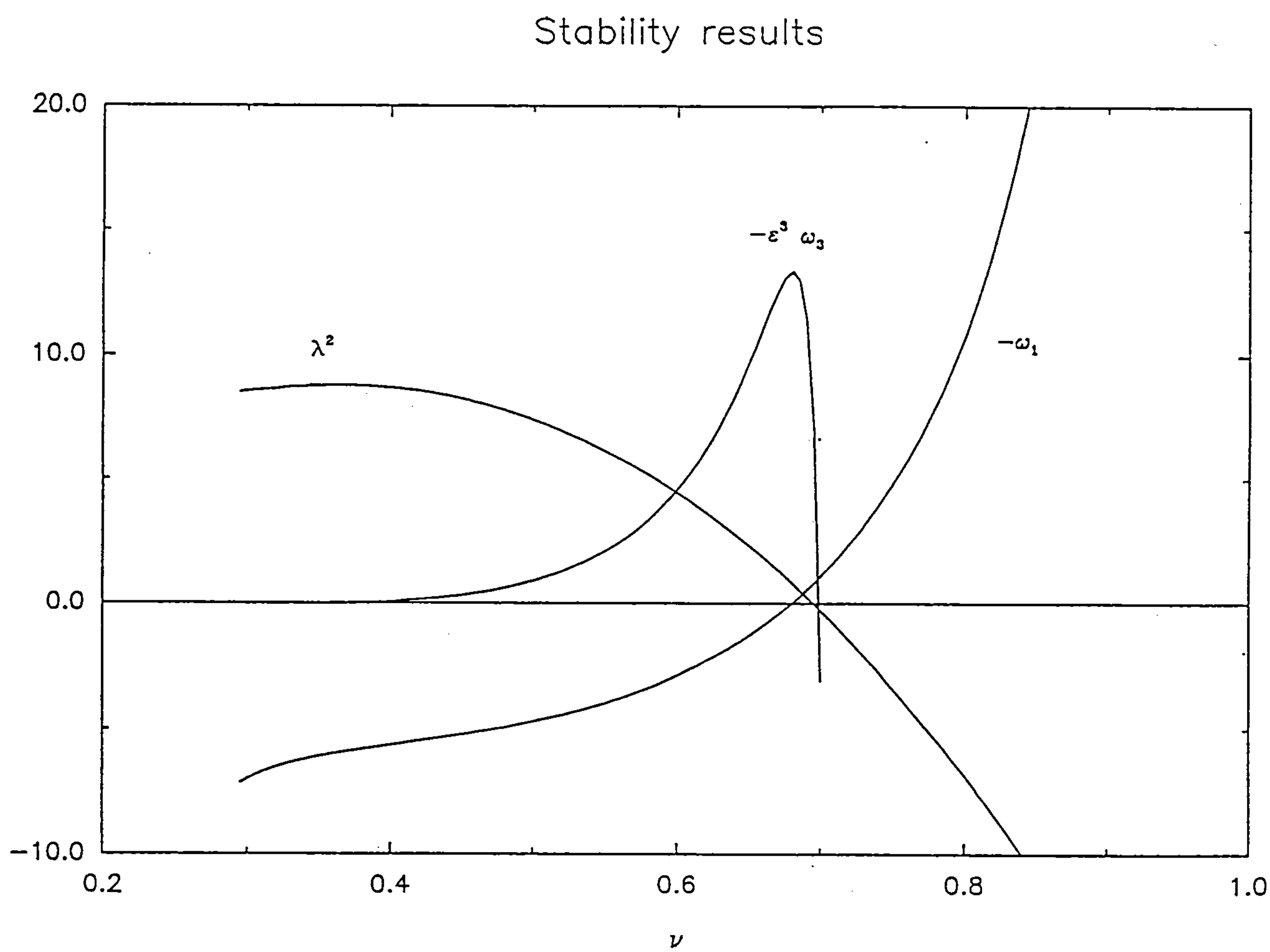
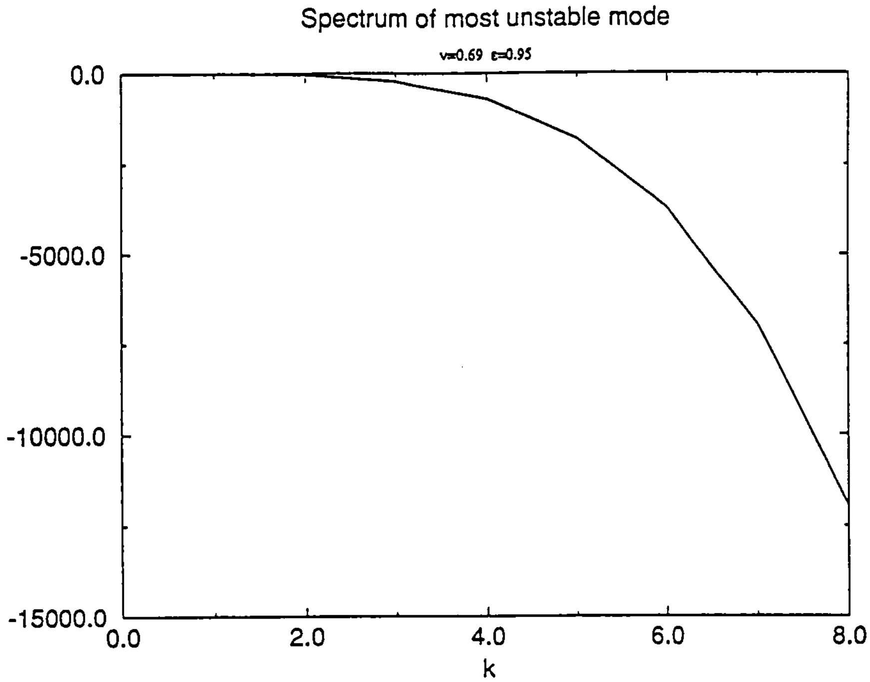
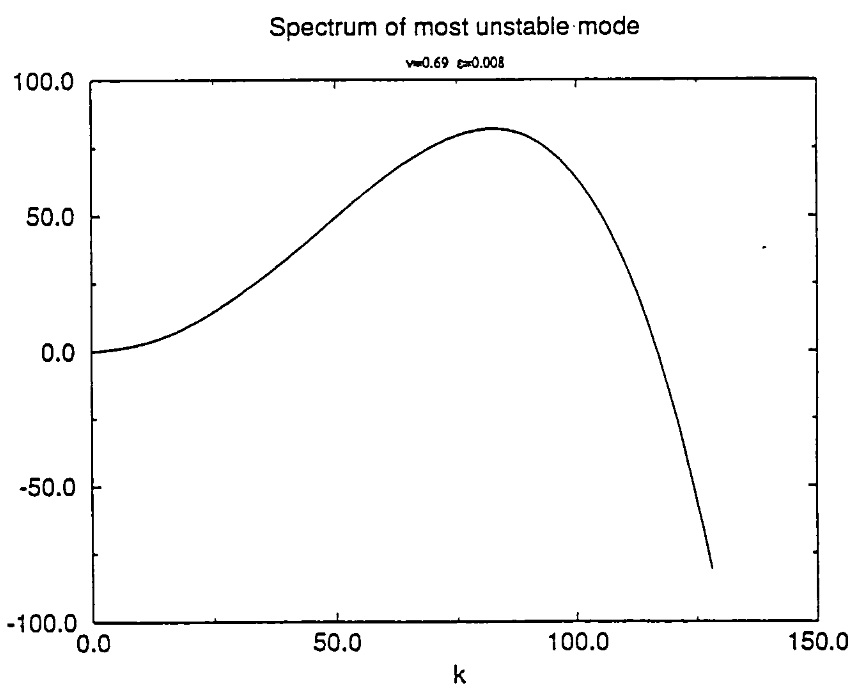


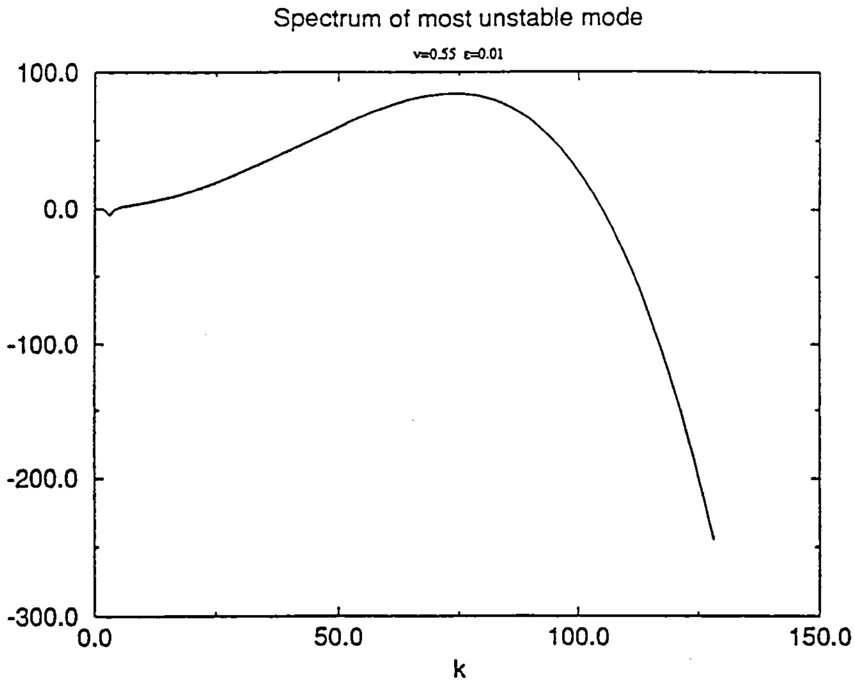
- **Figure 1** Linear stability characteristics, $\epsilon = 0.025$. Structural stability is achieved when λ^2 and $-\omega_3$ (from the fourth derivative terms) are positive. If we stop at $O(k^2)$, a small structural stability window is obtained when λ^2 and $-\omega_1$ are positive, as seen in the figure. Inclusion of higher order terms extends the structural stability window significantly.



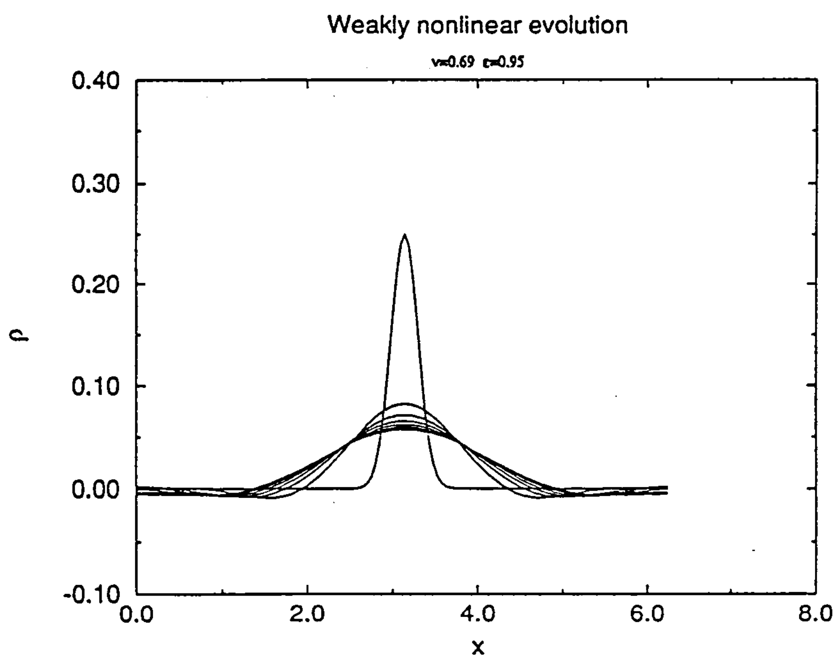
• **Figure 2** The most unstable eigenvalue computed from the exact dispersion relation (8.14) or equivalently (4.12) (the difference is a scaling factor due to a change of variables). The parameters are $\nu = 0.69$, $\epsilon = 0.95$ corresponding to a stability window where both the $O(k^2)$ and $O(k^4)$ terms are dissipative.



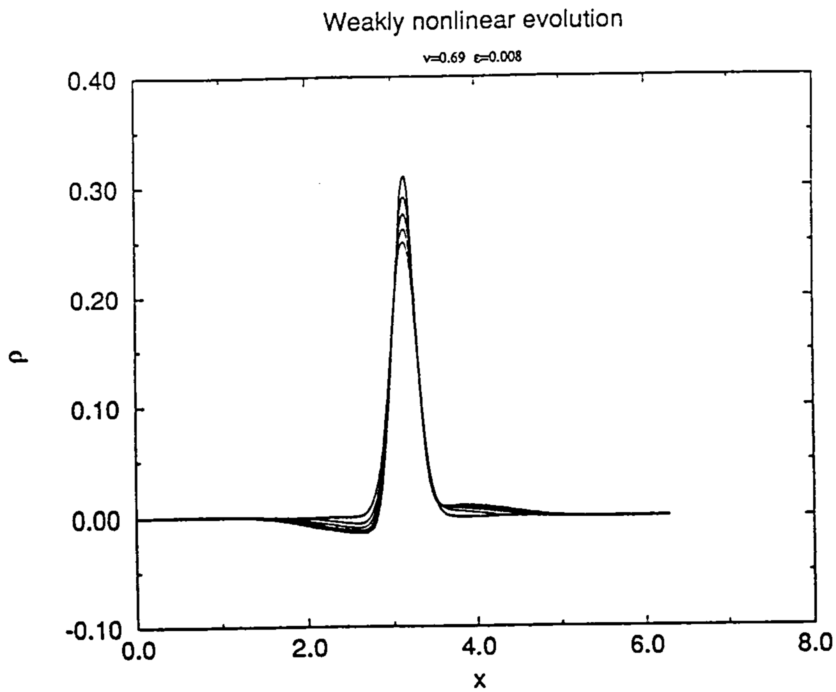
• **Figure 3** As in Figure 2, but $\epsilon = 0.008$. For this ϵ a large but finite band of unstable modes is obtained.



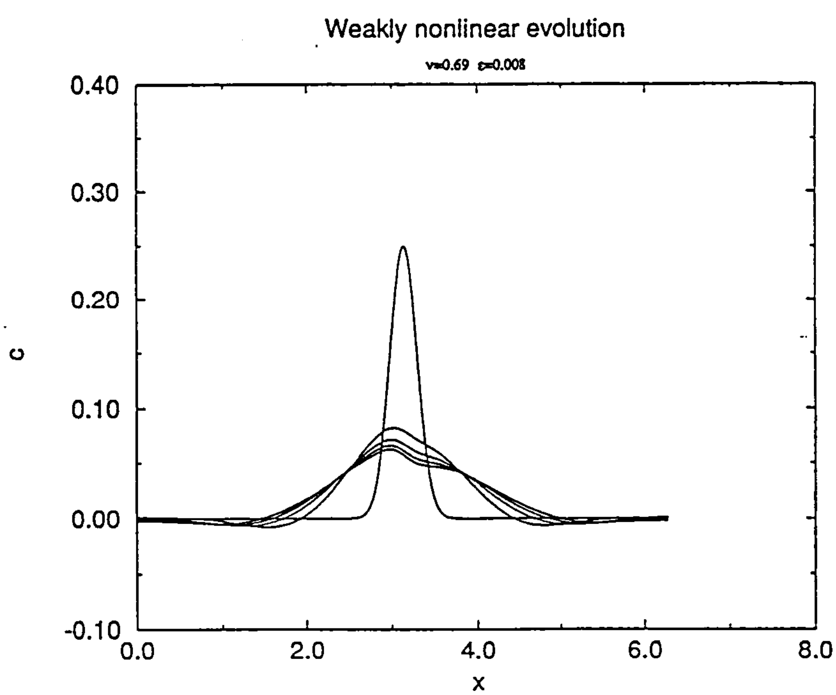
• **Figure 4** As in Figure 2, but $\nu = 0.55$, $\epsilon = 0.01$. For this ν the $O(k^2)$ term is unstable and the $O(k^4)$ term is dissipative. There is a small band of stable modes near $k = 0$ and a large band of unstable modes.



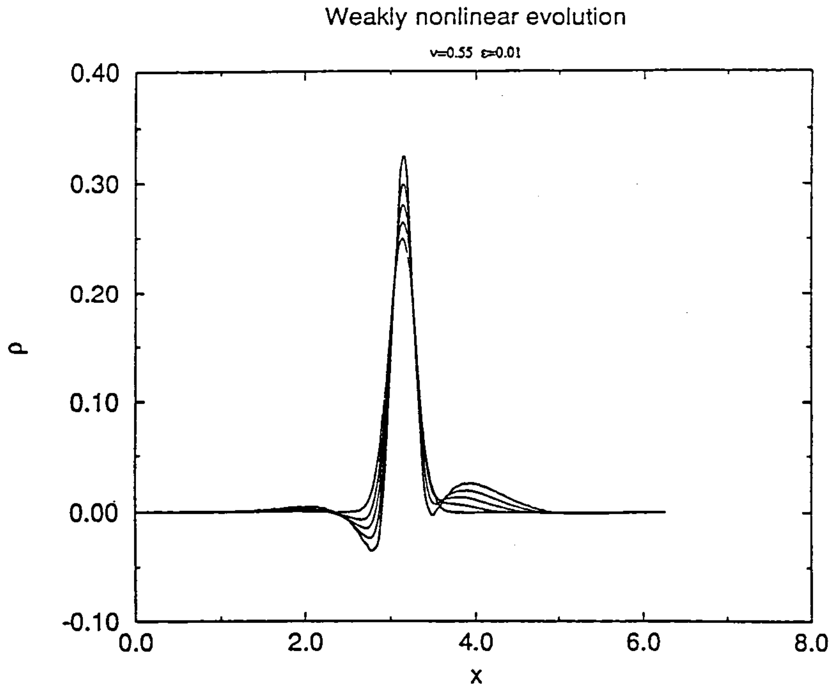
• **Figure 5** Solution of the nonlinear system (8.3), (8.4). The initial condition for ρ and c is a Gaussian of amplitude 0.25 centered at the midpoint of the domain. The evolution of ρ is shown for $\nu = 0.69$, $\epsilon = 0.95$. The corresponding c decays fast to a steady state (equal to the mean of the initial condition) by the end of the computation. The spectrum corresponding to this run is shown in Figure 2.



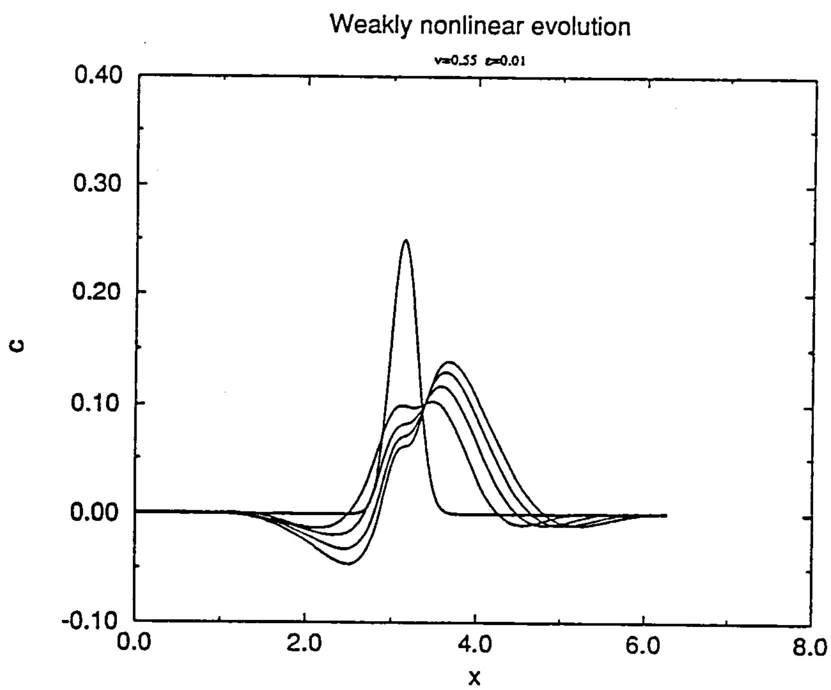
- **Figure 6a** As in Figure 5, the evolution of ρ for $\epsilon = 0.008$. The spectrum is shown in Figure 3.



- **Figure 6b** As in Figure 6a but evolution of c.



• **Figure 7a** As in Figure 5, evolution of ρ for $\nu = 0.55$, $\epsilon = 0.01$. The spectrum is shown in Figure 4.



• **Figure 7b** As in Figure 7a but evolution of c.

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We study the long-wave, modulational, stability of steady periodic solutions of the Kuramoto-Sivashinsky equation. The analysis is fully nonlinear at first, and can in principle be carried out to all orders in the small parameter, which is the ratio of the spatial period to a characteristic length of the envelope perturbations. In the linearized regime we recover a high-order version of the results of Frisch, She and Thual, [1], which shows that the periodic waves are much more stable than previously expected.

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